Now we turn to solutions to the 46th Ukrainian Mathematical Olympiad 2006, Final Round, given at [2009 : 23–24].

1. (V.V. Plakhotnyk) Prove that for any rational numbers a and b the graph of the function

$$f(x) \;=\; x^3 - 6abx - 2a^3 - 4b^3\,, \quad x \in \mathbb{R}$$

has exactly one point in common with the x-axis.

Solved by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA; Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's version.

We will make use of the following result proved at the end: $x^3-3px+2q$ vanishes exactly once for $x \in \mathbb{R}$ if and only if $q^2 > p^3$ or $p = q = 0$.

Here, $p = 2ab$ and $q = -(a^3 + 2b^3)$, hence $q^2 > p^3$ can be rewritten as $(a^3 + 2b^3)^2 > 8a^3b^3$, that is, $(a^3 - 2b^3)^2 > 0$. This is certainly true if $a^3 \neq 2b^3$. However, $a^3 = 2b^3$ cannot occur if a, $b \neq 0$, since otherwise the number 2 would be the cube of a nonzero rational number, which is impossible (if $m^3 = 2n^3$ for positive integers m and n, then a contradiction arises: the exponent of 2 in the standard factorization is a multiple of 3 on the left but not on the right).

Since $p = q = 0$ when $a = b = 0$, the condition $q^2 > p^3$ or $p = q = 0$ is satisfied for all rational numbers a and b, and the result follows.

We now prove the result used above. Let $P(x) = x^3 - 3px + 2q$. If $p \leq 0$, then for a, $b \in \mathbb{R}$ with $a \neq b$,

$$\frac{P(a) - P(b)}{a - b} \;=\; a^2 + ab + b^2 - 3p \;>\; 0\,,$$

hence P is increasing on $\mathbb{R}$ and vanishes only once.

If $p > 0$, then P is increasing on $(-\infty, -\sqrt{p})$ and $(\sqrt{p}, \infty)$ and decreasing on $(-\sqrt{p}, \sqrt{p})$. An easy calculation gives $P(-\sqrt{p}) = 2(q + p\sqrt{p})$ and $P(\sqrt{p}) = 2(q - p\sqrt{p})$, so that $P(-\sqrt{p}) > P(\sqrt{p})$ and we have that $P(-\sqrt{p}) \cdot P(\sqrt{p}) = 4(q^2 - p^3)$. It follows that if $q^2 < p^3$, then P vanishes once in $(-\sqrt{p}, \sqrt{p})$ as well as in $(-\infty, -\sqrt{p})$ and in $(\sqrt{p}, \infty)$.

If $0 < p^3 < q^2$, then $P(-\sqrt{p})$ and $P(\sqrt{p})$ have the same sign and P vanishes only once. Lastly if $p^3 = q^2 \neq 0$, then

$$P(x) \;=\; \left(x - \frac{q}{p}\right)^2 \left(x + \frac{2q}{p}\right)\,,$$

and P vanishes twice. The required result follows from these observations.

5. (O.O. Kurchenko) Prove that for any real numbers x and y

$$|\cos x| \;+\; |\cos y| \;+\; |\cos(x+y)| \;\geq\; 1\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give the argument of Bataille.

Let $v = e^{-2ix}$ and $w = e^{2iy}$. Then,

$$|1+v| \;=\; \left|e^{-ix}(e^{ix}+e^{-ix})\right| \;=\; \left|e^{-ix}(2\cos x)\right| \;=\; 2|\cos x|$$

and similarly,

$$|1+w| \;=\; 2|\cos y|$$

and

$$|v+w| \;=\; \left|e^{i(y-x)}(e^{i(x+y)}+e^{-i(x+y)})\right| \;=\; 2|\cos(x+y)|\,.$$

Now, using the Triangle Inequality, we obtain

$$\begin{aligned} 2 &= |(1+v)+(1+w)-(v+w)| \\ &\le |1+v|+|1+w|+|v+w| \\ &= 2\big(|\cos x|+|\cos y|+|\cos(x+y)|\big)\,, \end{aligned}$$

and the result follows.

6. (T.M. Mitelman) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\left(x^3+y^3\right) \;=\; x^2f(x) \;+\; yf(y^2)$$

for all real numbers x and y.

Solution by Michel Bataille, Rouen, France.

The solutions are the functions $f_m(x) = mx$, where m a real number.

It is readily checked that these functions satisfy the identity. We now show that there are no other solutions. To this aim, let f be any solution. Taking $x = y = 0$ in the identity yields $f(0) = 0$; also, with only $y = 0$, we obtain $f(x^3) = x^2f(x)$, and with only $x = 0$, we obtain $f(y^3) = yf(y^2)$. Thus, for all real numbers x,

$$f(x^3) \;=\; x^2f(x) \;=\; xf(x^2)\,.$$

From the identity we now obtain $f(x^3+y^3) = f(x^3)+f(y^3)$. Since any real number is the cube of a real number, it follows that

$$f(a+b) \;=\; f(a)+f(b)$$

for all real numbers a and b.

Consequently, f is odd (take $b = -a$), and $f(na) = nf(a)$ if $n \in \mathbb{Z}$ and $a \in \mathbb{R}$. Substituting $x+1$ and $x-1$ for x and y in the identity, we obtain on the one hand

$$f((x+1)^3+(x-1)^3) = f(2x^3+6x) = 2f(x^3)+6f(x) = 2x^2f(x)+6f(x)\,,$$